# The $q/(q-1)$ channel

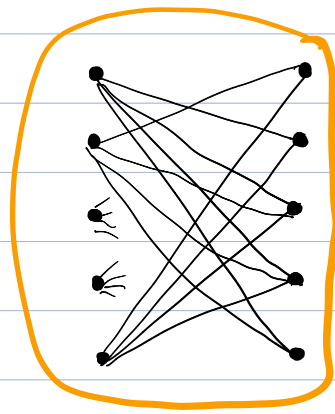
**Input alphabet:** $\{x_1, x_2, \ldots, x_q\} = X$

**Output alphabet:** $\{x_1, x_2, \ldots, x_q\}$

**Characteristics:**

$\Pr[(x_i, x_j) > 0]$ iff $i \neq j$.

When $x_i$ is transmitted, any symbol other than $x_i$ can be received.



The 5/4 channel.

**Code:** $C \subseteq X^n$, block length $= n$.

$C$ is an $\ell$-list decoding code for the $q/(q-1)$ channel if for every output word $\sigma' \in X^n$:

$$|\{\sigma \in C : \sigma \text{ is compatible with } \sigma'\}| \leq \ell.$$

$$\mathrm{Cap}(q,\ell) = \limsup_{n \to \infty} \max_{C} \frac{1}{n} \log_2 (m/\ell)$$

(Elias 1988)

Every set of $q-1$ output words is compatible with some input codeword.

$$\Downarrow$$

$$Cap(q, q-2) = 0$$

# Today

## Fredman-Komlós bound

$$\underbrace{\frac{1}{\sqrt{q}} e^{-q} \lesssim}_{\text{probabilistic argument}} Cap(q, q-1) \underbrace{\lesssim \sqrt{q}\, q\, e^{-q}}_{\text{density argument + entropy argument}}$$

probabilistic argument ✗

density argument + entropy argument ✓

$$0.212 \simeq \frac{1}{4}\log_2 \frac{9}{5} \leq Cap(3, 2) \leq \log_2 \frac{3}{2} \simeq 0.585$$

trifference

$$0.0473 \lessapprox \frac{1}{3}\log\frac{32}{29} \le \mathrm{Cap}(4,3) \le 6/19 \lessapprox 0.3158$$

w. Dalai + Guruswami

## Long lists

$$\frac{1}{q} \lesssim \mathrm{cap}(q, q\ln q) \lesssim 1/q$$

**Question:** How big must $\ell$ be so that

$$\frac{1}{q^c} \lesssim \mathrm{cap}(q,\ell)\ ?$$

**Answer:** $\ell \geq \frac{1}{5} q \ln q$

Because,
$$\mathrm{cap}(q, \varepsilon q \ln q) \lesssim \exp(-c\, q^{1-5\varepsilon})$$

w. Bhandari

# Graph Covering

$G_1, G_2, \ldots, G_t$ : bipartite graphs on $[n]$.

$$G_1 \cup G_2 \cup \ldots \cup G_t = K_n$$

$$\Downarrow$$

$$t \geq \log_2 n$$

## Two proofs (?)

From each $G_i$ delete one side at random

$$\underbrace{\Pr[v \text{ survives}]}_{\text{these events are disjoint}} = 1/2^t$$

$$\Downarrow$$

$$n \cdot \frac{1}{2^t} \leq 1$$

$$\Downarrow$$

$$t \geq \log n.$$

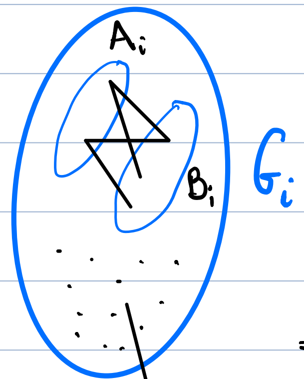
Pick a vertex $v$ of $K_n$ uniformly at random.

Let $X_1, X_2, \ldots, X_t$ be the colours of $v$ in $G_1, \ldots, G_t$.

$$\begin{aligned}\log_2 n &= I[v \mid X_1 \ldots X_t] \\ &\leq H[X_1 \ldots X_t] \\ &\leq \sum_i H[X_i] \\ &\leq t\end{aligned}$$

# Hansel's refinement

$$G_1 \cup G_2 \cup \dots \cup G_t = K_n$$




$n - n_i$ isolated

$n_i$ = number of non-isolated vertices in $G_i$.

$$\Downarrow$$

$$\sum_{i=1}^{t} n_i \geq n \log_2 n$$

Let $l_x$ be the total number of non-isolated occurrences of vertex $x$.

Delete vertices in $A_i$ or $B_i$ from $G_i$ (each with prob. $1/2$)

$$\Pr[x \text{ survives}] = 2^{-l_x}.$$

$$\Downarrow$$

$$\sum_x 1/2^{l_x} \leq 1$$

$$\Downarrow$$

$$\sum_{i=1}^{t} n_i = \sum_x l_x \geq n \log_2 n$$

Pick $v \in [n]$ at random.

Let $X_i = \begin{cases} 0 & \text{if } v \in A_i \\ 1 & \text{if } v \in B_i \\ 0/1 & \text{otherwise.} \end{cases}$

$$\begin{aligned} \log_2 n &= I[v : X_1 \dots X_t] \\ &\leq \sum_{i=1}^{t} I[v : X_i] \\ &= \sum H[X_i] - H[X_i / v] \\ &\leq \sum_i (1 - (n - n_i)/n) \\ &= \sum_i n_i / n \end{aligned}$$

## The probabilistic proof

What can we do if the $G_i$ are not bipartite? In each step, we retain an independent set $Y_i$ of the graph $G_i$. Let $p_{ix} = \Pr[x \in Y_i]$. Then,

$$\sum_x \prod_i p_{ix} \leq 1$$

$$\Downarrow$$

$$\left(\prod_x \prod_{i=1}^{t} p_{ix}\right)^{1/n} \leq n$$

$$\sum_i \underbrace{\sum_x \log_2 p_{ix}^{-1}}_{} \geq \log_2 n$$

$$\check{H}(G) = \max_Y \sum_x \log_2 \Pr[x \in Y]$$

$$\Downarrow$$

$$\sum_{i=1}^{t} \check{H}(G_i) \geq \log_2 n$$

# The entropy proof

Pick a vertex $X$ uniformly at random and pick an independent set $Y_i$ in $G_i$ s.t. $x \in Y_i$.

$$I[X : Y_1 \dots Y_t] \geq \log_2 n$$

$$\wedge|$$

$$\sum_i \underbrace{I[X : Y_i]} \geq \log_2 n$$

$$\hat{H}(G) = \max_{(X,Y)} I[X : Y]$$

$X$ uniform, $Y$ an ind. set s.t. $X \in Y$.

$$\Downarrow$$

$$\sum_{i=1}^{t} \hat{H}(G_i) \geq \log_2 n$$

# Körner's graph entropy

$$\mathcal{H}(G) = \vec{\mathcal{H}}(G) = \hat{\mathcal{H}}(G)$$

**Subadditivity**

$$G = G_1 \cup G_2$$

$$\Downarrow$$

$$\mathcal{H}(G_1) + \mathcal{H}(G_2) \geq \mathcal{H}(G).$$

**Also,**

$$\mathcal{H}(G, P) = \lim_{n\to\infty} \frac{1}{n} \log_2 \chi^{\varepsilon}(G^n)$$

$\forall\, \varepsilon \in (0,1).$

The minimum chromatic number of a subgraph of $G^n$ that includes at least $\varepsilon$ of its mass.

# The Fredman-Komlós bound

$C \subseteq [q]^n$, $|C| = m$.

For every set of $q$ code words, there is a coordinate where they all differ.

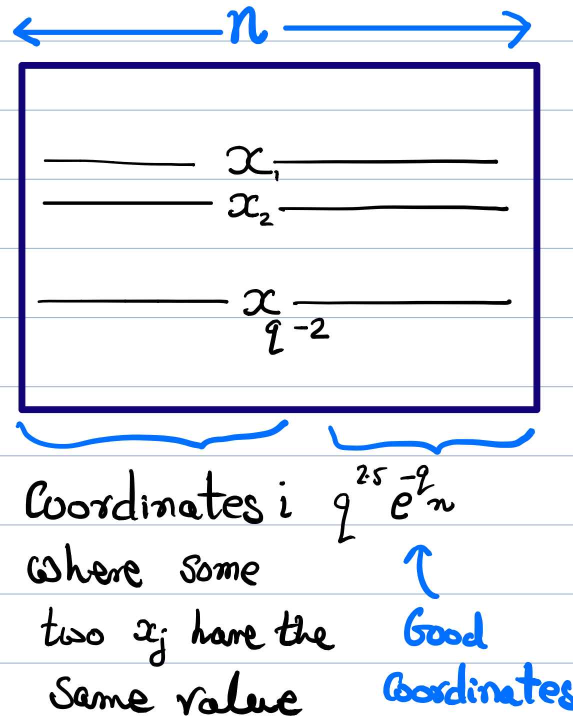
$$\Downarrow$$

$$n \geq \frac{e^q}{q\sqrt{q}} \log_2 m$$

$$\Downarrow$$

$$cap(q, q-1) \leq q^{1.5} e^{-q}$$

Pick $x_1, x_2, \ldots, x_{q-2} \in C$ at random without replacement.




Every pair of codewords in $C \setminus \{x_1, x_2, \ldots, x_{q-2}\}$ differs in some 'good' coordinate.

$$\Downarrow$$

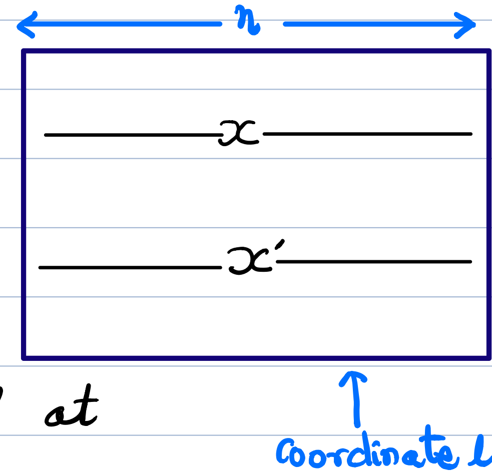
$$q^{2.5} e^{-q} n \geq \log_2 (m - q + 1)$$

$$\Downarrow$$

$$n \geq \frac{e^k}{q^{2.5}}$$

We can improve this by a factor $q$ by using Hansel's lemma. We will do this more careful calculation for the $4/3$ channel.

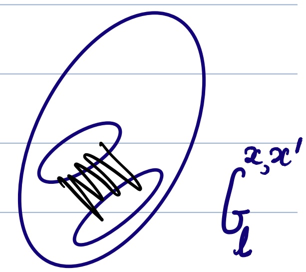
## The 4/3 channel




- Pick $x$ and $x'$ at random
- The codewords in $C \setminus \{x, x'\}$ must be separated from each other and from $x, x'$ at one of the coordinates.

$$\bigcup_{l=1}^{n} G_l^{(x,x')} = K_{m-2}$$

On average $G_l$ has $\simeq m \cdot \left(\frac{3}{8}\right)$ non-isolated vertices.




$$\Downarrow$$

$$n \cdot m \cdot \frac{3}{8} \geq (m-2) \log_2 (m-2)$$

$$\Downarrow$$

$$n \geq \left(\frac{8}{3}\right) \log m$$

$$\Downarrow$$

$$\mathrm{Cap}(4,3) \leq \frac{3}{8} = 0.375$$

The number of non-isolated vertices = Codewords that differ from both $x$ and $x'$ at position $l$.

Arikan: If $C$ has large rate, then there must be two code words that are close. In the coordinates where they agree, $G_\ell$ will be empty.

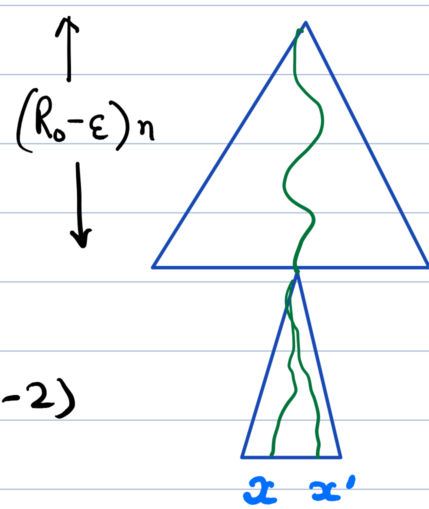
$$\Downarrow$$

$$Cap(4,3) \leq 0.3512$$

## W. Dalai + Guruswami

Idea: Examine the Plotkin bound more closely.

Suppose $|C| = 4^{R_0 n}$




If $x$ and $x'$ are chosen from a common subtree, then

$$\frac{3}{8}(1-R_0+\varepsilon) \geq \log_2(m-2)$$

$$\simeq 2R_0$$

$$\Downarrow$$

$$R_0 \leq \frac{3}{19}$$

$$\Downarrow$$

$$Cap(4,3) \leq \frac{6}{19} \simeq 0.3158$$

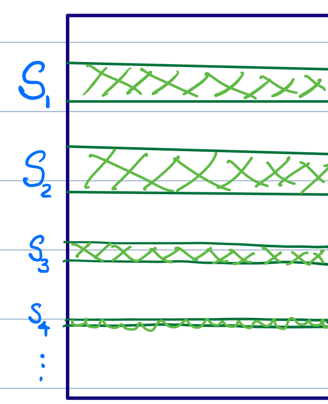
# Long lists

$$\mathrm{cap}(q, \varepsilon q \ln q) \lesssim \exp(-c q^{1-5\varepsilon})$$

$C \subseteq [q]^n$, $|C| = m$

Suppose $n \ll \exp(-c q^{1-5\varepsilon}) \log m$.

**Goal:** Show that $\exists S \subseteq C$ $|S| = \varepsilon q \ln q$ s.t. in each coordinate $S$ misses some symbol in $[q]$. If we regard coordinates as hash functions $h_i : C \to [q]$, then $h_i(S) \neq [q]$ for all $i \in [n]$.




- Pick a set of $q-2$ code words. In a typical column, these code words pick up $\sim q - \frac{q}{e}$ symbols. $\longrightarrow S_1$

- Pick a set of $\approx \frac{q}{e} - 2$ symbols. In a typical column, we expect $\sim (q/e)^{\exp(\frac{-1}{e})}$ symbols unpicked. $\longrightarrow S_2$

- And so on, ..., pick sets $S_3, S_4, \ldots$

## Difficulty

A typical columns.

- Argue that they will be few.

$$\lesssim \log m / \exp(c'q)$$

- Eliminate the atypical columns by restricting attention to a subcode.

$$C \rightsquigarrow C'$$

$$|C| = m \qquad |C'| = m' = m^{1-\varepsilon}$$

- This works up to a point!

$$\Downarrow$$

$$\mathrm{cap}(q, 1.58q) = \exp(-\Omega(q))$$

(w. Chakraborty, Raghunathan, Sasatte '06)

**Idea:**

- Replace $S_i$ by an ensemble of sets.
- Eliminate columns that are atypical for <u>any</u> set in the ensemble.
- A lot more columns to eliminate.
- Technical but elementary.

## Please see:

Siddharth Bhandari + Jaikumar Radhakrishnan. Bounds on the zero-error list-decoding capacity of the $q/(q-1)$ channel. ISIT '18

Email jaikumar@tifr.res.in for the latest version.

# Summary

- Hansel's lemma
- Graph entropy
- The Fredman-Komlós bound
- The 4/3 channel
- The q/(q-1) channel with big lists.

$\boxed{\text{Thank you!}}$

These notes were prepared for the talk on "The zero-error list-decoding capacity of the q/(q-1) channel." Communication, Control and Signal Processing Series (CCSP) at UMD on 11 Feb 2021.